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MOTION OF A ROCKET DURING THE BURNING PERIOD

by

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MOTION OF A ROCKET DURING THE BURNING PERIOD

Abstract

In this report the general equations of motion of a rocket are developed. These equations are then applied to a rocket which is subjected to an eccentric propelling force, one of the principal causes of inaccuracy in rocket firings. The equations are first specialized for the vacuum case of the non-rotating rocket. The results for this case are compared with those for a rotating rocket moving in a vacuum. A formula for the angular velocity necessary to reduce dispersion is given. Finally, aerodynamic forces are con - sidered and expressions for the yaw, angle of deviation and deflection are set down.

1. Introduction

The equations of motion which govern the flight of a symmetric rocket will be developed in a general way in order that they may be used for reference purposes, and used in the analysis of problems associated both with the burning period and the post-burning period.

After the equations have been set up, this report will be primarily concerned with the effect of an imperfect alignment of the propelling jet. The eccentricity of the propelling force is one of the main causes of inaccuracy in the firing of rockets.

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2. Notation

In setting up the equations of motion for a rocket the following symbols and conventions will be used:

$\underline{c}_1, \underline{c}_2, \underline{c}_3$ = unit vectors defining a right-handed Newtonian reference frame. The vector $\underline{c}_1$ points in the direction in which the rocket is launched.

$\underline{i}, \underline{j}, \underline{k}$ = unit vectors defining a right-handed moving reference system whose origin O is at the center of mass of the rocket. The vector $\underline{i}$ coincides with the longitudinal axis of the rocket. The vectors $\underline{j}$ and $\underline{k}$ are temporarily left undefined. Unless there is a statement to the contrary, all vectors are considered as resolved with respect to the moving axes.

$\underline{v} = (v_1, v_2, v_3)$ = velocity of O.

$\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ = angular velocity of rocket.

$\underline{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ = angular velocity of moving axes.

$\underline{F} = (F_1, F_2, F_3)$ = resultant propelling force at O.

$\underline{P} = (P_1, P_2, P_3)$ = resultant propelling moment about O.

$\underline{f} = (f_1, f_2, f_3)$ = resultant aerodynamic force at O.

$\underline{p} = (p_1, p_2, p_3)$ = resultant aerodynamic moment about O.

$\underline{g} = (g_1, g_2, g_3)$ = acceleration due to gravity.

m = mass of rocket.

A = axial moment of inertia of rocket.

B = transverse moment of inertia of rocket.

N = axial spin of rocket relative to moving axes.

$\underline{h} = \left[A(N+\Omega_1), B\,\Omega_2, B\,\Omega_3\right]$ = angular momentum of rocket about O. Since $\Omega_2 = \omega_2$, $\Omega_3 = \omega_3$, the angular momentum can be written

$$\underline{h} = (A\omega_1, B\omega_2, B\omega_3).$$

3. General Equations of Motion

In vector form, the fundamental equations of motion are

$$\frac{d(m\,\underline{v})}{dt} = \underline{F} + \underline{f} + \underline{g}; \quad \frac{d\underline{h}}{dt} = \underline{P} + \underline{p}.$$

Written out at length in scalar form these equations yield

$$(m^{\cdot}v_1) + m(\omega_2 v_3 - v_2\omega_3) = F_1 + f_1 + mg_1$$

$$(m^{\cdot}v_2) + m(\omega_3 v_1 - v_3\Omega_1) = F_2 + f_2 + mg_2$$

$$(m^{\cdot}v_3) + m(\Omega_1 v_2 - v_1\omega_2) = F_3 + f_3 + mg_3$$

$$(A\,\dot{\omega}_1) = P_1 + p_1$$

$$(B\dot{\omega}_2) + A\omega_3\omega_1 - B\omega_3\Omega_1 = P_2 + p_2$$

$$(B\dot{\omega}_3) + B\Omega_1\omega_2 - A\omega_1\omega_2 = P_3 + p_3$$

where the dot denotes differentiation with respect to the time t. The above equations can be expressed more conveniently and more compactly by introducing the complex quantities

$$v_c = v_2 + iv_3 \qquad \omega_c = \omega_2 + i\omega_3$$

$$F_c = F_2 + iF_3 \qquad f_c = f_2 + if_3 \qquad g_c = g_2 + ig_3$$

$$P_c = P_2 + iP_3 \qquad p_c = p_2 + ip_3.$$

In terms of these quantities the equations of motion reduce to

$$(m\dot{v}_1) + m(\omega_2 v_3 - v_2\omega_3) = F_1 + f_1 + mg_1$$

$$(m\dot{v}_c) + im(\Omega_1 v_c - v_1\omega_c) = F_c + f_c + mg_c$$

$$(A\dot{\omega}_1) = P_1 + p_1$$

$$(B\dot{\omega}_c) - i\,(A\omega_1 - B\Omega_1)\,\omega_c = P_c + p_c.$$

For certain parts of the development which follows it is necessary to know the moving axes components of a vector $\underline{r}$ which is fixed with respect to the inertial frame. If this vector is written

$$\underline{r} = r_1\,\underline{i} + r_2\underline{j} + r_3\underline{k}$$

the components can be found by noting that $\dot{\underline{r}} = 0$ which leads to

$$\dot{r}_1 + \omega_2 r_3 - \omega_3 r_2 = 0$$

$$\dot{r}_2 + \omega_3 r_1 - \Omega_1 r_3 = 0$$

$$\dot{r}_3 + \Omega_1 r_2 - \omega_2 r_1 = 0$$

or, if $r_c = r_2 + ir_3$ is introduced, to

$$\dot{r}_1 + \omega_2 r_3 - \omega_3 r_2 = 0$$

$$\dot{r}_c + i(\Omega_1 r_c - r_1\omega_c) = 0.$$

These equations associated with the proper initial conditions for the particular vector $\underline{r}$ determine its moving axes components r_1, r_2, r_3.

It will be interesting later to find the yaw δ, (the angle between $\underline{i}$ and $\underline{v}$); the angle of deviation θ, (the angle between $\underline{v}$ and $\underline{c}_1$); and the angle φ between $\underline{i}$ and $\underline{c}_1$. These angles can be found provided $\underline{v}$ and $\underline{\Omega}$ are known. The yaw is given by

$$\tan \delta = \left|\frac{\underline{i} \times \underline{v}}{\underline{i} \cdot \underline{v}}\right| = \left|\frac{v_2 \underline{k} - v_3 \underline{i}}{v_1}\right| = \frac{\sqrt{v_2^2 + v_3^2}}{v_1} = \left|\frac{v_c}{v_1}\right| .$$

If $\underline{r} = \underline{c}_1$ the angle of deviation is given by

$$\tan \theta = \left|\frac{\underline{c}_1 \times \underline{v}}{\underline{c}_1 \cdot \underline{v}}\right| = \frac{\sqrt{(r_1v_2 - r_2v_1)^2 (r_1v_3 - r_3v_1)^2 + (r_2v_3 - r_3v_2)^2}}{r_1v_1 + r_2v_2 + r_3v_3}$$

which for sufficiently small r_2, r_3, v_2, v_3 can be replaced by

$$\tan \theta \sim \sqrt{\frac{(r_1v_2 - r_2v_1)^2 + (r_1v_3 - r_3v_1)^2}{v_1v_1}} = \left|\frac{(r_1v_2 - r_2v_1)}{r_1v_1} + \frac{(r_1v_3 - r_3}{r_1v_1}\right|$$

$$= \left|\frac{v_2}{v_1} - \frac{r_2}{r_1} + \frac{v_3 i}{v_1} - \frac{r_3 i}{r_1}\right| = \left|\frac{v_c}{v_1} - \frac{r_c}{r_1}\right| .$$

The angle φ is given by

$$\tan \varphi = \left|\frac{\underline{i} \times \underline{c}_1}{\underline{i} \cdot \underline{c}_1}\right| = \frac{\sqrt{r_2^2 + r_3^2}}{r_1} = \left|\frac{r_c}{r_1}\right| .$$

For the usual burning period the above angles are generally small; the tangents of these angles can then of course be replaced by the angles themselves.

4. Aerodynamic Forces

The analysis of the aerodynamic forces acting on the rocket will be based on the assumptions and methods introduced by Nielsen and Synge in their very important paper.* In accordance with these authors, it is assumed that each component of $\underline{f}$ is a function of ρ, c, v_1, v_2, v_3, ω_1, ω_2, ω_3; where ρ is the density of the atmosphere and c is the velocity of sound. If v_2, v_3, ω_2, ω_3 are small the components f_2 and f_3 can be replaced by the approximations

* K. L. Nielsen and J. L. Synge, On the Motion of a Spinning Shell, reproduced by the Ballistic Research Laboratory by permission of the National Research Council of Canada, 1943.

$$f_2 = a_1 v_2 + b_1 v_3 + c_1 \omega_2 + d_1 \omega_3$$

$$f_3 = a_2 v_2 + b_2 v_3 + c_2 \omega_2 + d_2 \omega_3$$

in which constant terms do not appear since if $v_2 = v_3 = \omega_2 = \omega_3 = 0$ then f_2 and f_3 must vanish. The coefficients in the approximations are functions of ρ, c, v_1, ω_1. The cross force $f_c = f_2 + if_3$ is thus

$$f_c = (a_1 + ia_2)\, v_2 + (b_1 + ib_2)\, v_3 + (c_1+ic_2)\omega_2+(d_1+id_2)\omega_3$$

or using $v_c = v_2 + iv_3$, $\omega_c = \omega_2 + i\omega_3$

$$f_c = \alpha_1 v_c + \beta_1 \overline{v}_c + \gamma_1 \omega_c + \delta_1 \overline{\omega}_c$$

where the coefficients are complex functions of ρ, c, v_1, ω_1.

It is assumed now that the rocket is symmetric in the sense that a rotation of the rocket through the angle $\psi = 2\pi/n$, where n is an integer greater than 2, about its longitudinal axis, restores the rocket to its original position. For such a rocket the coefficients of the conjugate terms in f_c are zero. Synge and Biot have shown this in the following way. For any given motion we can consider an alternate motion defined by velocity vectors which result from rotating rigidly the velocity vectors $\underline{v}$ and $\underline{\omega}$ through the angle, ψ, about the axis of the rocket. If this alternate motion is considered as referred to the same axis used to describe the actual motion, then $v_c e^{i\psi}$, $\omega_c e^{i\psi}$ are the new velocities. From the symmetry of the rocket the new cross force must be $f_c e^{i\psi}$ and according to the assumptions $f_c e^{i\psi}$ must be equal to

$$f_c e^{i\psi} = \alpha_1 v_c e^{i\psi} + \beta_1 \overline{v}_c e^{-i\psi} + \gamma_1 \omega_c e^{i\psi} + \delta_1 \overline{\omega}_c e^{-i\psi} .$$

If the original expression for f_c is substituted in this equation, it is found that

$$2i\beta_1 \sin\psi = 0 \qquad 2i\delta \sin\psi = 0$$

$$\beta_1 = 0 \qquad \delta_1 = 0$$

and hence

$$f_c = \alpha_1 v_c + \gamma_1 \omega_c .$$

By the same analysis

$$p_c = \alpha_2 v_c + \gamma_2 \omega_c .$$

The cross force and the cross moment can be written in the more explicit forms

$$f_c = \left[\theta_1(v_1,\omega_1) + i\theta_2(v_1,\omega_1)\right] v_c + \left[\theta_3(v_1,\omega_1) + i\theta_4(v_1,\omega_1)\right] \omega_c$$

$$p_c = \left[\theta_5(v_1,\omega_1) + i\theta_6(v_1,\omega_1)\right] v_c + \left[\theta_7(v_1,\omega_1) + i\theta_8(v_1,\omega_1)\right] \omega_c$$

It is also assumed that f_1 and p_1 are functions of ρ, c, v_1, v_2, v_3; ω_1, ω_2, ω_3; but that the effects of v_2, v_3; ω_2, ω_3 on f_1 and p_1 can be entirely neglected. Consider then f_1 and p_1 as functions of ρ, c, v_1, ω_1 only. From the dimensional standpoint, it is convenient to define quantities K_{DA}, and K_A by means of the equations

$$f_1 = \rho v_1^2 d^2 K_{DA}$$

$$p_1 = \rho v_1 \omega_1 d^4 K_A$$

where d is the diameter of the rocket and the K's are dimensionless functions. f_1 is the axial drag and p_1 is the spin decelerating moment.

The θ's which appear in f_c and p_c can be interpreted as follows. Suppose

$$\omega_1 = \omega_2 = \omega_3 = 0, \text{ then}$$

$$f_c = \left[\theta_1(v_1,0) + i\theta_2(v_1,0)\right] (v_2+iv_3)$$

$$p_c = \left[\theta_5(v_1,0) + i\theta_6(v_1,0)\right] (v_2 + iv_3).$$

If in addition $v_3 = 0$, then $f_2 = \theta_1(v_1,0)v_2$; $f_3 = \theta(v_1,0)v_2$; $p_2 - \theta_5(v_1,0)v_2$; $p_3 = \theta_6(v_1, 0)v_2$; but for this case the motion is planar and f_3 and p_2 must be zero. To meet these requirements set $\theta_2 = \omega_1 \theta_2{*}(v_1, \omega_1)$; $\theta_5 = \omega_1\theta_5{*}(v_1, \omega_1)$.

Then for $\omega_1 = \omega_2 = \omega_3 = 0$, $v_3 \neq 0$,

$$|f_c| = \theta_1(v_1, 0) |V_c|$$

$$|p_c| = \theta_6(v_1, 0) |V_c| .$$

Dimensional considerations then lead to

$$\theta_1 = \rho v_1 d^2 K_N \qquad \theta_6 = \rho v_1 d^3 K_M.$$

The quantity $\theta_1 v_c$ is the cross force due to cross velocity, on the normal force, and it has the same orientation as the cross velocity. The quantity $\theta_6 v_c$ is the cross torque due to cross velocity, or the restoring torque.

Suppose now that $v_1 = 0, \omega_2 = \omega_3 = 0$. Then

$$|f_c| = \omega_1\theta_2{*}(0, \omega_1)|v_c|$$

$$|p_c| = \omega_1\theta_5^{*} (0, \omega_1)|v_c|$$

and from dimensional considerations

$$\theta_2 = \rho\omega_1 d^3 K_F \qquad \theta_5 = \rho\omega_1 d^4 K_T.$$

The quantity $\theta_2 v_c$ is the Magnus cross force due to cross velocity, and the quantity $\theta_5 v_c$ is the Magnus cross torque due to cross velocity.

If $v_2 = v_3 = 0;\ \omega_1 = 0,$ then

$$f_c = \left[\theta_3(v_1,0) + i\theta_4(v_1, 0)\right](\omega_2 + i\omega_3)$$

$$p_c = \left[\theta_7(v_1 0) + i\theta_8(v_1,0)\right](\omega_2 + i\omega_3)$$

and if in addition $\omega_2 = 0$, then $f_2 = -\theta_4(v_1, 0)\omega_3;\ f_3 = \theta_3(v_1,0)\omega_3;$ $p_2 = -\theta_8(v_1, 0)\omega_3;\ p_3 = \theta_7(v_1,0)\omega_3;$ but for this case the motion is planar. Therefore f_3 and p_2 must be zero. To meet these requirements set $\theta_3 = \omega_1\theta_3{*}(v_1, \omega_1);\ \theta_8 = \omega_1\theta_8{*}(v_1,\omega_1)$. Then for $v_2 = v_3 = 0,\ \omega_1 = 0,$

$$\left|f_c\right| = \theta_4(v_1,0)\left|\omega_c\right|$$

$$\left|p_c\right| = \theta_7(v_1,0)\left|\omega_c\right|$$

Dimensional theory then leads to

$$\theta_4 = \rho v_1 d^3 k_s \qquad \theta_7 = \rho v_1 d^4 k_H.$$

The quantity $\theta_4\ \omega_c$ is the cross force due to cross spin and the quantity $\theta_7\omega_c$ is the cross torque due to cross spin.

Suppose now that $v_2 = v_3 = 0,\ v_1 = 0.$ Then

$$\left|f_c\right| = \omega_1\theta_3{}^*(0, \omega_1)\left|\omega_c\right|$$

$$\left|p_c\right| = \omega_1\theta_8{}^*(0, \omega_1)\left|\omega_c\right|$$

and again from dimensional considerations

$$\theta_3 = \rho\omega_1 d^4 k_{xF} \qquad \theta_8 = \rho\omega_1 d^5 k_{xT}$$

The quantity $\theta_3\omega_c$ is the Magnus cross force due to cross spin, and the quantity $\theta_8\omega_c$ is the Magnus cross torque due to cross spin.

5. Vacuum Case

It is instructive to study first the trajectory of a non-rotating rocket which is subjected to a propelling force $\underline{F}(t)$ which acts parallel to the longitudinal axis of the rocket at a distance of ε units from that axis. In this section the effect of such an imperfect alignment of the propelling jet on a rocket moving in a vacuum will be investigated.

Let the moving axes be fixed in the rocket and take the $\underline{k}$ axis so that it coincides in direction with the propelling movement vector. Then $\underline{f} = 0$; $\underline{p} = 0$; $F_1 = F_1(t)$; $F_2 = 0$; $F_3 = 0$; $P_1 = 0$; $P_2 = 0$; and $P_3 = F_1\varepsilon$. For the particular case under consideration, it is convenient to introduce $\xi = v_1+iv_2$; $\sigma = g_1 + ig_2$ and $\eta = \omega_2 + i\omega_3$. Taking into account the fact that the mass of the rocket varies with the time, the equations of motion are

$$(\dot{A\omega_1}) = 0$$

$$(\dot{Bn}) - i(A-B)\omega_1 n = iF_1\varepsilon$$

$$(\dot{m\xi}) + im\omega_3\xi + mv_3(\omega_2 - i\omega_1) = F_1 + m\sigma$$

$$(\dot{mv_3}) + m(\omega_1 v_2 - v_1\omega_2) = mg_3$$

If time is measured from the instant at which the rocket emerges from the launching tube, the initial velocities are $\underline{v} = v_o\underline{i}$; $\underline{\omega} = 0$ for $t = 0$.

The equations for the components of $\underline{g}$ are

$$\dot{\sigma} + i\omega_3\sigma + g_3(\omega_2 - i\omega_1) = 0$$

$$\dot{g}_3 + \omega_1 g_2 - \omega_2 g_1 = 0$$

The initial values of the components of $\underline{g}$ depend upon the choice of the inertial frame. This frame is chosen so that $\underline{c}_1$ coincides with the axis of the launching tube and $\underline{c}_3$ is horizontal. Let μ be the angle of elevation of the tube. The components of $\underline{g}$ referred to the in frame are then $(-g\sin\mu$, $- g\cos\mu$, $0)$. Suppose that the plane determined

by $\underline{i}$ and $\underline{F(t)}$ at $t = 0$ makes an angle ν with the plane determined by $\underline{c}_1$ and $\underline{c}_2$. With respect to the moving axes, the initial components of $\underline{g}$ are then $(-g \sin\mu, -g\cos\mu\cos\nu, -g\cos\mu\sin\nu)$.

From the first equation of motion $\omega_1 = 0$ and hence from the second $B(\omega_2 + i\omega_3) = i\int_0^t F_1 \varepsilon \, dt = i\,h(t)$ so that $\omega_2 = 0$; $\omega_3 = h/B$. The components of g are given by

$$g_3 = \text{cons't.} = -g\cos\mu\sin\nu$$

and

$$\dot{\sigma} + i\,\omega_3\sigma = 0$$

from which

$$\sigma = g_1 + ig_2 = C_1 e^{-i\int_0^t \omega_3 dt} = C_1 e^{-is(t)}$$

where $\quad s(t) = \int_0^t \omega_3 dt = \varphi$; and $C_1 = -g\sin\mu - ig\cos\mu\cos\nu$.

The remaining equations reduce to

$$v_3 = \frac{-g\cos\mu\sin\nu}{m}\int_0^t m\,dt$$

and

$$(m\dot{\xi}) + i\,m\,\omega_3\,\xi = F_1 + mC_1 e^{-is(t)}$$

from which

$$m\,\xi = m(v_1 + iv_2) = \int_0^t F_1 e^{\,i\left[s(T)-s(t)\right]} dT + \left[C_1\int_0^t m\,dt + m_o v_o\right] e^{-i\,s(t)}$$

where m_o is the initial mass and v_o is the initial velocity.

If $\underline{r}$ is a vector which is fixed with respect to the inertial frame, the components r_1, r_2, r_3 of this vector with respect to the moving axes are given by the equations

$$\dot{r}_1 - \omega_3 r_2 = 0$$

$$\dot{r}_2 + \omega_3 r_1 = 0$$

$$\dot{r}_3 = 0$$

If $\zeta = r_1 - ir_2$ is introduced, these equations become:

$$\dot{\zeta} - i\,\omega_3 \zeta = 0$$

$$\dot{r}_3 = 0$$

or

$$\zeta = c_2 e^{is(t)}$$

$$r_3 = \text{const.}$$

from which v_1, v_2 and v_3 are easily determined. Applying these equations to $\underline{c}_1, \underline{c}_2$ and $\underline{c}_3$ we find

	Initial components		
$\underline{c}_1$:	$(1, 0, 0)$	$\zeta = e^{is(t)}$	$r_3 = 0$
$\underline{c}_2$:	$(0, \cos\nu, \sin\nu)$	$\zeta = -i\cos\nu\, e^{is}$	$r_3 = \sin\nu$
$\underline{c}_3$:	$(0, -\sin\nu, \cos\nu)$	$\zeta = i\sin\nu\, e^{is}$	$r_3 = \cos\nu$

If $\underline{r}$ is a unit vector, the component of $\underline{v}$ along $\underline{r}$ is $\underline{v}\cdot\underline{r} = v_1 r_1 + v_2 r_2 + v_3 r_3 = R_e(\xi\zeta) + v_3 r_3$. Therefore the components of $\underline{v}$ with respect to the inertial frame are:

$$V_1 = \frac{1}{m}\int_0^t F_1 \cos\left[s(T)\right] dT - \frac{g \sin\mu}{m}\int_0^t m\, dt + \frac{m_o v_o}{m}$$

$$V_2 = \frac{\cos\nu}{m}\int_0^t F_1 \sin\left[s(T)\right] dT - \frac{g \cos\mu}{m}\int_0^t m\, dt$$

$$V_3 = -\frac{\sin\nu}{m}\int_0^t F_1 \sin\left[s(T)\right] dT \quad .$$

Writing $M(t) = \int_0^t \frac{dt}{m}$ the integrals of the above equations can be written

$$X = \int_0^t \left[M(t) - M(T)\right] F_1 \cos\left[s(T)\right] dT - g \sin\mu \int_0^t \left[M(t)-M(T)\right] m(T)dT$$
$$+ m_o v_o M(t)$$

$$Y = \cos\nu \int_0^t \left[M(t)-M(T)\right] F_1 \sin\left[s(T)\right] dT - g \cos\mu \int_0^T \left[M(t)-M(T)\right] m(T)dT$$

$$Z = -\sin\nu \int_0^t \left[M(t) - M(T)\right] F_1 \sin\left[s(T)\right] dT.$$

Thus the restricted problem of this section is solved in terms of quadratures.

The deflection, and the angle which the axis of the rocket makes with the line of departure (the $\underline{c}_1$ axis) are given respectively by the equations

$$d = \sqrt{Y^2 + Z^2}; \quad \varphi = s(t).$$

The angle of deviation, and the yaw are given respectively by the equations

$$\tan\theta = \sqrt{V_2^2 + V_3^2}/V_1 \; ; \quad \tan\delta = \sqrt{V_2^2 + V_3^2}/V_1.$$

All of the above quantities except φ involve gravity terms. These terms, however, can generally be neglected.

If m, F, B, and ε are regarded as constants, and if gravity is neglected the deflection d is

$$d = \frac{F_1}{m}\int_0^t\int_0^t \sin \frac{F_1 \varepsilon t^2}{2B}\, dt\, dt;$$

the angle φ is

$$\varphi = \int_0^t\int_0^t \frac{F_1 \varepsilon}{B}\, dt\, dt = \frac{F_1 \varepsilon\, t^2}{2B}$$

and the angle of deviation is

$$V_1 \tan \theta = \frac{F_1}{m}\int_0^t \sin \frac{F_1 \varepsilon t^2}{2B}\, dt.$$

Approximate expressions for d and Θ are

$$d = \frac{F_1{}^2 \varepsilon t^4}{24\ m\ B}$$

$$\theta = \frac{F_1^2 \varepsilon t^3}{6\ m\ BV}\quad .$$

The yaw is the difference $\varphi - \theta$. Thus it is seen that the quantities d, φ, θ & are approximately directly proportional to the eccentricity factor ε.

6. Rotating Rocket Vacuum Case

If a rocket is caused to rotate about its longitudinal axis with an angular velocity ω_1, it is not unreasonable to suppose that ω_1 will tend to overcome any eccentric action of the propelling forces. In order to investigate this suspected stabilizing effect of ω_1, consider now the motion of a rotating rocket. For the time being suppose that the motion takes place in a vacuum, and suppose that the effect of gravity can be neglected. Let the propelling force $\underline{F}$ and the propelling moment P be arbitrary functions of t. Let the moving axes be fixed in the rocket, and take $\underline{k}$ on the plane determined by $\underline{P}$ and $\underline{i}$. Choose the inertial frame so that $\underline{c}$ coincides with the axis of the launching tube, and $\underline{c}_3$ is horizontal. Let the rocket emerge from the tube where t = 0. and assume that for t = 0, $\underline{v} = v_o \underline{i}$, $\underline{\omega} = \omega_o \underline{i}$.

The equations of motion are

$$(A\dot{\omega}_1) = P_1$$

$$(B\dot{\omega}_c) - i(A-B)\omega_1\omega_c = iP_3$$

$$(m\dot{v}_1) + m\,(\omega_2 v_3 - v_2\omega_3) = F_1$$

$$(m\dot{V}_c) + i\omega_1\, mv_c - imv_1\omega_c = F_c\,.$$

From the first equation

$$\omega_1 = \frac{1}{A}\int_0^t P_1 dt + \omega_o.$$

Regarding $B\omega_c$ as dependent variable in the second equation, the solution of this equation is easily found to be

$$\omega_c = \frac{1}{B}\, e^{-i\int_0^t k\omega_1 dt}\int_0^t e^{i\int_0^t k\omega_1 dt} P_3\, dt$$

where $k = 1 - A/B$. Since $|F_c|$ and P_3 are small in comparison with F_1 and P_1 respectively, the quantity $\omega_2 v_3 - v_2\omega_3$ can be neglected in the third equation which then yields

$$v_1 = \frac{1}{m}\int_0^t F_1\, dt + v_o.$$

The solution of the fourth equation can now be found. It is

$$v_c = \frac{1}{m}\, e^{-i\int_0^t \omega_1 dt}\int_0^t e^{i\int_0^t \omega_1 dt} F_c dt + \frac{1}{m}\, e^{i\int_0^t \omega_1 dt}\int_0^t e^{i\int_0^t \omega_1 dt} mv_1\omega_c dt$$

The moving axes components of a vector $\underline{r}$ which is fixed with respect to the inertial frame are given by

$$\dot{r}_1 + \omega_2 r_3 - \omega_3 r_2 = 0$$

$$\dot{r}_c + i\omega_1 r_c - ir_1\omega_c = 0.$$

If $\underline{r} = \underline{c}_1$ and $\omega_2 r_3 - \omega_3 r_2$ is neglected,

$$r_1 = \text{cons't.}$$

$$\frac{r_c}{r_1} = i\, e^{-i\int_0^t \omega_1 dt} \int_0^t e^{i\int_0^t \omega_1 dt}\, \omega_c dt.$$

Expressions for the yaw, the angle of deviation and the deflection can now be set down. The yaw δ is determined by

$$\tan\delta = \left|\frac{v_c}{v_1}\right| \frac{1}{mv_1} \left| \int_0^t e^{i\int_0^t \omega_1 dt} F_c dt - \int_0^t \frac{mv_1}{B}\, e^{i\int_0^t (1-k)\omega_1 dt} \int_0^t P_3 e^{i\int_0^t k\omega_1 dt}\, dt\, dt \right|$$

and the angle of deviation is determined by

$$\tan\theta = \left|\frac{v_c}{v_1} - \frac{r_c}{r_1}\right| = \frac{1}{mv_1} \left| \int_0^t e^{i\int_0^T \omega_1 dT} F_c dT \right.$$

$$\left. + \int_0^t \frac{\left[m(t)v_1(t) - m(T)v_1(T)\right]}{B(T)}\, e^{i\int_0^T (1-k)\omega_1 dT} \int_0^T e^{i\int_0^s k\omega_1 ds} P_3 ds dT \right|$$

The component of $\underline{v}$ perpendicular to $\underline{c}_1$ is $v_1 \tan\theta$. The deflection d is then $\int_0^t v_1 \tan\theta\, dt$.

Suppose that m, B, F_1, ε are constants; and suppose $F_c=0$, $\omega_1=0$, $P_3 = F_1\varepsilon$. Then

$$v_1 \tan\theta = \frac{F_1^2\varepsilon}{mB} \int_0^t (t-T) \int_0^T ds\, dT = \frac{F_1^2 \varepsilon t^3}{6m\,B}$$

and

$$d = \frac{F_1^2\, \varepsilon t^4}{24mB}$$

These results check with those found in the previous section.

Suppose now that m, B, A, F_1, ε are constants; $F_c=0$, $P_3 = F_1\varepsilon$, but that $\omega_1 \neq 0$. Then if $\int_0^t \omega_1 dt = s(t)$

$$v_1 \tan\theta = \frac{F_1^2\varepsilon}{mB} \left| \int_0^t (t-T) e^{i(1-k)s(T)} \int_0^T e^{iks(x)} dx\, dT \right| .$$

The integral which appears above can be appraised as follows. Integration of the inner integral by parts leads to

$$v_1 \tan\theta = \frac{F_1^2\varepsilon}{mB} \left| \begin{array}{l} \displaystyle\int_0^t \frac{(t-T)}{ik\omega_1} e^{is(T)} dT - \int_0^t \frac{(t-T)e}{ik\omega_0}^{i(1-k)s(T)} dT \\ \displaystyle + \frac{1}{ik} \int_0^t (t-T) e^{i(1-k)s(T)} \int_0^T \frac{e^{iks(x)}}{\omega_1^2} \lambda\, dx\, dT \end{array} \right| \qquad (A)$$

Noting that the absolute magnitude of a sum is equal to or less than the sum of the absolute magnitudes of the terms of the sum,

$$v_1 \tan\theta \leq \frac{F_1^2\varepsilon}{mB} \left| \begin{array}{l} \displaystyle\int_0^t \frac{(t-T)dT}{k\omega_1} + \int_0^t \frac{(t-T)dT}{k\omega_0} \\ \displaystyle + \frac{1}{k} \int_0^t (t-T) \int_0^T \frac{\lambda}{\omega_1^2} dx\, dT \end{array} \right|$$

Integration then yields

$$v_1 \tan\theta \leq \frac{F_1^2\varepsilon\, t^2}{mBk\omega_0} .$$

If this result is compared with the corresponding result for the case of no rotation, viz., $v_1 \tan \theta = F_1^2 \varepsilon t^3/6mB$, it can be easily seen that if the angular velocity is to be effective in reducing the angle of deviation then

$$\frac{1}{k\omega_o} \leqq \frac{t}{6}$$

or

$$\omega_o \geqq \frac{6}{kt} .$$

It should be noted that if the above procedure for the appraisal of $v_1 \tan \theta$ is applied to the expression (A) a better estimate can be obtained, namely

$$v_1 \tan \theta \leqq \frac{t^2}{2k\omega_o} + \frac{2t}{k\omega_o^2} + \frac{\lambda t^2}{k^2\omega_o^3} .$$

7. Rotating Rocket Subjected to Aerodynamic Forces.

The motion of a rocket during the burning period will now be investigated under less restrictive conditions than have hitherto been imposed. Three of the aerodynamic forces will be admitted into the equations of motion of a spinning rocket, namely, the drag $= \rho d^2 K_D v^2 = J_D v^2$, the lift $= \rho d^2 K_L v^2 \sin\delta = J_L v^2 \sin\delta$, and the righting moment $= \rho d^3 K_M v^2 \sin = J_M v^2 \sin\delta$, where δ is the yaw. Only these forces are admitted because they are the most important. It is to be noted, however, that even if all the aerodynamic forces are admitted, the equations of motion can be solved in the manner discussed below.

Let the inertial frame be chosen as above. Let the moving axes be fixed in the rocket with $\underline{i}$ coinciding with the axis of the rocket and $\underline{k}$ in the plane determined by $\underline{i}$ and $\underline{P}$. Suppose that the rocket emerges from the tube when $t = 0$; and that for $t = 0$ $\underline{v} = v_o \underline{i}$, $\omega = \omega_o \underline{i}$.

Using the terminology introduced earlier, the axial drag f_1 is equal to

$$f_1 = J_L v^2 \sin^2\delta - J_D v^2 \cos\delta$$

$$= J_L v^2 \tan^2\delta - J_D v_1^2 \sin\delta$$

the normal force, or cross force due to cross velocity is

$$|\theta_1|\,|v_c| = J_L v^2 \sin\delta \cos\delta + J_D v^2 \sin\delta$$

$$|\theta_1| = J_L v \cos\delta + J_D v = J_L v_1 + J_D v_1 \sec\delta$$

and the righting moment, or cross torque due to cross velocity is

$$|\theta_6|\,|v_c| = J_L v^2 \sin\delta$$

$$|\theta_6| = J_M v = J_M v_1 \sec\delta$$

Therefore

$$f_c = -(J_L v_1 + J_D v_1 \sec\delta) v_c$$

$$p_c = \quad i J_M v_1 \sec\delta \; v_c.$$

The equations of motion are

$$(m\dot{v}_1) + m(\omega_2 v_3 - v_2\omega_3) = F_1 + (J_L \tan^2\delta - J_D \sec\delta) v_1^2 + mg_1$$

$$(m\dot{v}_c) + i\omega_1 m v_c - i m v_1 \omega_c = F_c - (J_L + J_D \sec\delta) v_1 v_c + mg_c$$

$$(A\dot{\omega}_1) = P_1$$

$$(B\dot{\omega}_c) + ik\omega_1 B\omega_c \quad = iP_3 + i(J_M \sec\delta)\, v_1 v_c$$

$$k = 1 - \frac{A}{B}$$

and the equations for the determination of the components of $\underline{g}$ with respect to the moving axes are

$$\dot{g}_1 + \omega_2 g_3 - \omega_3 g_2 = 0$$

$$\dot{g}_c + i\omega_1 g_c - i g_1 \omega_c = 0.$$

No attempt will be made here to solve the above equations of motion in a general manner. Instead, the analysis will be confined to the simplified equations which result from assuming $\omega_2 v_3 - v_2 \omega_3$, and the effect of gravity both negligible; and assuming m constant and δ small. The equations then become

$$\dot{v}_1 = \frac{F_1}{m} - \frac{J_D}{m} v_1^2$$

$$\dot{v}_c + i\omega_1 v_c - i v_1 \omega_c = \frac{F_c}{m} - \left(\frac{J_L + J_D}{m}\right) v_1 v_c$$

$$\dot{\omega}_1 = \frac{P_1}{A}$$

$$\dot{\omega}_c + ik\omega_1 \omega_c = \frac{iP_3}{B} + \frac{i}{B} J_M v_1 v_c.$$

The first and third equations of this set can easily be solved. The solutions are

$$v_1 = \sqrt{\frac{F_1}{J_D}} \coth \left(\frac{\sqrt{J_D F_1}}{m} t - b\right),$$

where the constant of integration b is determined by

$$v_o = \sqrt{\frac{F_1}{J_D}} \coth b;$$

and

$$\omega_1 = \int_0^t \frac{P_1}{A} dt + \omega_o.$$

The second and fourth equations are not easy to handle because of the variable coefficients. It is interesting to note, however, that they can be solved explicitly by assuming $\omega_1 = av_1$ and introducing the new variable s defined by $ds = v_1 dt$.

The assumption $\omega_1 = av_1$ is perhaps a dubious one; but it is not pointless to solve the equations on the basis of this assumption for if a = 0 is substituted in the solution so obtained it reduces to the solution for the non-spinning case. If there $\omega_1 = av_1$, and $s = \int_0 v_1 dt$ are used, the second and fourth equations become

$$\frac{dv_c}{ds} + \left(ia + \frac{J_L + J_D}{m}\right) v_c - i\omega_c = \frac{F_c}{mv_1}$$

$$\frac{d\omega_c}{ds} + ika\omega_c - \frac{iJ_M v_c}{B} = \frac{iP_3}{Bv_1}$$

which can be solved readily. The solutions are

$$v_c = \frac{1}{\alpha_1 - \alpha_2}\left[e^{\alpha_1 s}\int_0^t e^{-\alpha_1 s}\left(\frac{\alpha_1 F_c}{m} + \frac{ikaF_c}{m} - \frac{P_3}{B}\right)dt - e^{\alpha_2 s}\int_0^t e^{-\alpha_2 s}\left(\frac{\alpha_2 F_c}{m} + \frac{ikaF_c}{m} - \frac{P_3}{B}\right)dt \right]$$

$$\omega_c = \frac{1}{\alpha_1 - \alpha_2}\left[e^{\alpha_1 s}\int_0^t e^{-\alpha_1 s}\left[\frac{iJ_M F_c}{Bm} + \left\{\frac{i(J_L+J_D)}{m} - a + i\alpha_1\right\}\frac{P_3}{B}\right]dt - e^{\alpha_2 s}\int_0^t e^{-\alpha_2 s}\left[\frac{iJ_M F_c}{Bm} + \left\{\frac{i(J_L+J_D)}{m} - a + i\alpha_2\right\}\frac{P_3}{B}\right]dt \right]$$

in which

$$\alpha_1 = \frac{-\ell_1 + \sqrt{\ell_1^2 - 4\ell_2}}{2}$$

$$\alpha_2 = \frac{-\ell_1 - \sqrt{\ell_1^2 - 4\ell_2}}{2}$$

where

$$\ell_1 = ia(k+1) + \left(\frac{J_L + J_D}{m}\right)$$

$$\ell_2 = \frac{J_M}{B} - ka^2 + ika\left(\frac{J_L + J_D}{m}\right).$$

The above solutions can be used to calculate the yaw, the angle of deviation and the deflection. For example, take the special case for which $\omega_1 = 0$; $F_1 = \text{cons't}$; $F_c = 0$; $P_1 = 0$; $P_3 = F_1\varepsilon$; $J_L = J_D = 0$. The roots α_1 and α_2 are then

$$\alpha_1 = i\sqrt{\frac{J_M}{B}} \qquad \alpha_2 = -i\sqrt{\frac{J_M}{B}}$$

and hence

$$v_c = -\frac{F_1\varepsilon}{\sqrt{J_M B}}\int_0^t \sin\sqrt{\frac{J_M}{B}}\left[s(t) - s(T)\right] dT$$

$$\omega_c = \frac{F_1\varepsilon}{B}\int_0^t \cos\sqrt{\frac{J_M}{B}}\left[s(t) - s(T)\right] dT.$$

Using the above equations, it is easy to verify that $\omega_2 = 0$; $v_3 = 0$. The motion is therefore planar, and it can be assumed without loss of generality that the rocket moves in the $\underline{c}_1\ \underline{c}_2$ plane.

The yaw is given by

$$\tan\delta = \left|\frac{v_c}{v_1}\right| = \frac{F_1\varepsilon}{v_1\sqrt{J_M B}}\int_0^t \sin\frac{m}{2F_1}\sqrt{\frac{J_M}{B}}\left[\left(\frac{F_1 t}{m} + v_0\right)^2 - \left(\frac{F_1 T}{m} + v_0\right)^2\right] dT.$$

This can be written in a simpler and more usable form by introducing

$$G = \frac{F_1}{m} \qquad \mu = \sqrt{\frac{J_M}{B}} \qquad k_1^2 = \frac{B}{m}$$

$$\sqrt{\frac{\mu}{G}}\, v_1 = \sqrt{\frac{\mu}{G}}\,(Gt + v_o) = \sqrt{\pi}\, z$$

$$\sqrt{\frac{\mu}{G}}(G\,T + v_o) = \sqrt{\pi} x$$

The expression for the yaw then becomes

$$\tan\delta = \frac{\varepsilon}{\mu k_1^2 z}\int_{z_0}^{z} \sin\frac{\pi}{2}(z^2 - x^2)\,dx.$$

The angle φ is given by $\tan\varphi = |r_c/r_1|$ or (see last section) by $\tan\varphi = \left| i\int_0^t \omega_c dt \right| = \left| \sqrt{\pi/\mu G}\int_{z_0}^{z} \omega_c dz \right|$. Now

$$\omega_c = \frac{iG\varepsilon}{k_1^2}\sqrt{\frac{\pi}{\mu G}}\int_{z_0}^{z} \cos\frac{\pi}{2}(z^2 - x^2)\,dx.$$

An integration then yields

$$\tan\varphi = \frac{\varepsilon\pi}{2\mu k_1^2}\left[\left\{\int_{z_0}^{z} \cos\frac{\pi}{2}x^2\,dx\right\}^2 + \left\{\int_{z_0}^{z} \sin\frac{\pi}{2}x^2\,dx\right\}^2\right].$$

For a more detailed analysis of the last results see the report by I. Bowen, L. Davis, L. Blitzer: The Effect of Fin Size, Burning Time, and Projector Length on the Accuracy of Rockets, NDRC, CIT-JPC3.

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General equations of motion of a symmetric rocket are developed and applied to an eccentric propelling force. Effects of the resulting imperfect alignment of propelling jet are discussed, and formulas are given for angular velocity necessary to reduce dispersion. The equations are also specialized for the vacuum case of nonrotating rockets and the results are compared with those for a rotating rocket in a vacuum. Aerodynamic forces are considered and expressions for the yaw, angle of deviation, and deflection are computed.

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